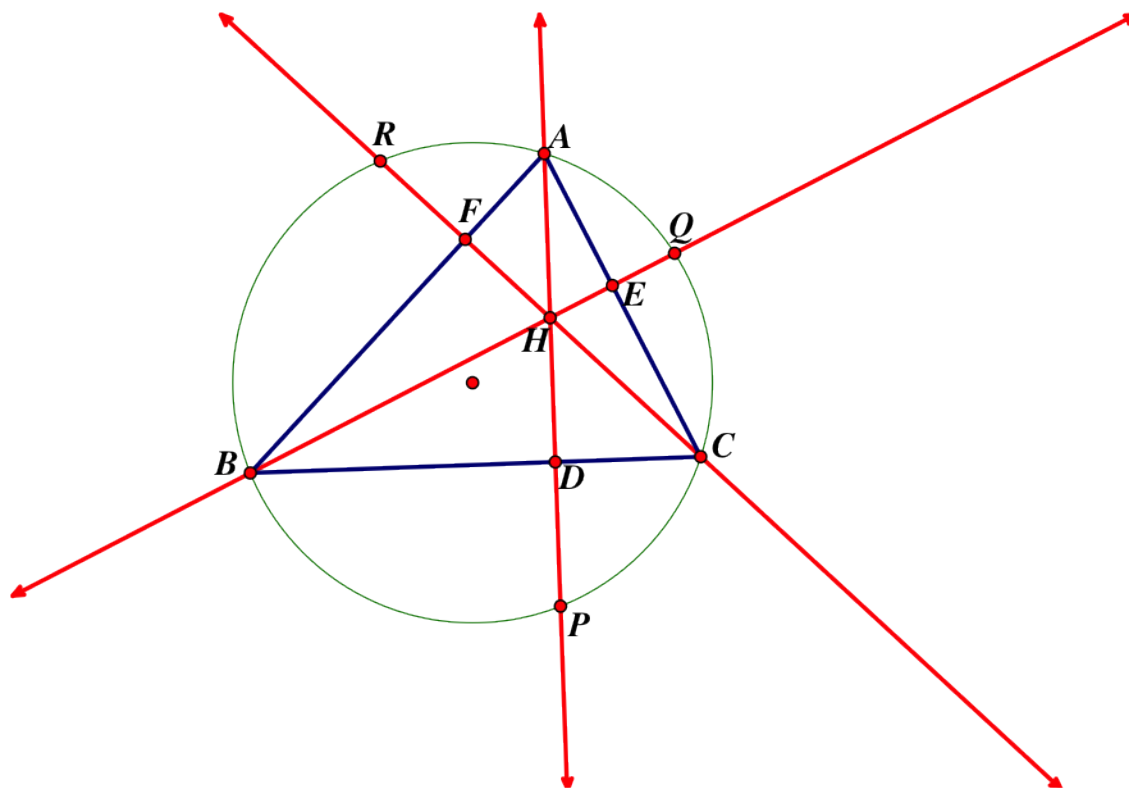


Assignment 8 Problem 5 - Write-up: On An Interesting Relationship Between Altitudes and Circumcircles of Acute Triangles

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Problem 5 of Exploration #8 starts with a rather innocuous construction: (paraphrased) Construct an acute triangle $\triangle ABC$, its circumcircle, and its extended altitudes. We present the construction here.



It then prompts an investigation of the value of the following expression:

$$\frac{AP}{AD} + \frac{BQ}{BE} + \frac{CR}{CF} \quad (\star)$$

It is clear that the expression will be at least three, as each of the three individual ratios is larger than 1. Upon investigation of the particular construction in Geometer's Sketchpad, one calculates

$$\begin{aligned} AP &= 12.02 \text{ cm} \\ AD &= 7.87 \text{ cm} \\ BQ &= 11.82 \text{ cm} \\ BE &= 9.86 \text{ cm} \\ CR &= 11.22 \text{ cm} \\ CF &= 8.80 \text{ cm} \\ \frac{AP}{AD} + \frac{BQ}{BE} + \frac{CR}{CF} &= 4.00 \end{aligned}$$

In order to prove this result, it may perhaps be easiest to rewrite the expression (\star) as follows.

$$\begin{aligned} &\frac{AP}{AD} + \frac{BQ}{BE} + \frac{CR}{CF} \\ &= \frac{AD + DP}{AD} + \frac{BE + EQ}{BE} + \frac{CF + FR}{CF} \\ &= 1 + \frac{DP}{AD} + 1 + \frac{EQ}{BE} + 1 + \frac{FR}{CF} \\ &= 3 + \frac{DP}{AD} + \frac{EQ}{BE} + \frac{FR}{CF} \end{aligned}$$

Proving that (\star) equals 4 is thus reduced to proving

$$\frac{DP}{AD} + \frac{EQ}{BE} + \frac{FR}{CF} = 1$$

Now, this ratio seems similar to Ceva's Theorem and other geometric proofs involving areas of triangles, so we will look for ratios that we can set up using triangles with equal bases.

Consider

$$\frac{A(\triangle BHC)}{A(\triangle ABC)} = \frac{DH}{AD}$$

since DH and AD are the heights of $\triangle BHC$ and $\triangle ABC$, respectively, and the bases of the triangles are each BC . Similarly, we can recognize

$$\frac{A(\triangle AHC)}{A(\triangle ABC)} = \frac{EH}{BE}, \quad \frac{A(\triangle AHB)}{A(\triangle ABC)} = \frac{FH}{CF}$$

Therefore,

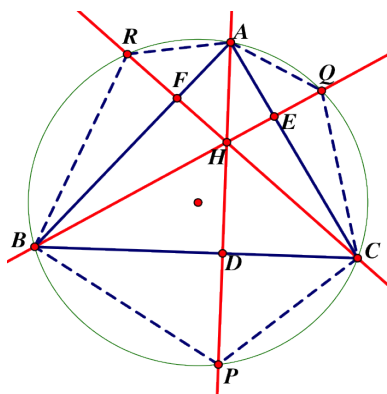
$$\begin{aligned} \frac{DH}{AD} + \frac{EH}{BE} + \frac{FH}{CF} &= \frac{A(\triangle BHC)}{A(\triangle ABC)} + \frac{A(\triangle AHC)}{A(\triangle ABC)} + \frac{A(\triangle AHB)}{A(\triangle ABC)} \\ &= \frac{A(\triangle BHC) + A(\triangle AHC) + A(\triangle AHB)}{A(\triangle ABC)} = \frac{A(\triangle ABC)}{A(\triangle ABC)} \\ &= 1 \end{aligned}$$

Now, compare the following expression, which we just proved equal to 1, and (★):

$$1 = \frac{DH}{AD} + \frac{EH}{BE} + \frac{FH}{CF}, \quad \&$$

$$\frac{DP}{AD} + \frac{EQ}{BE} + \frac{FR}{CF}$$

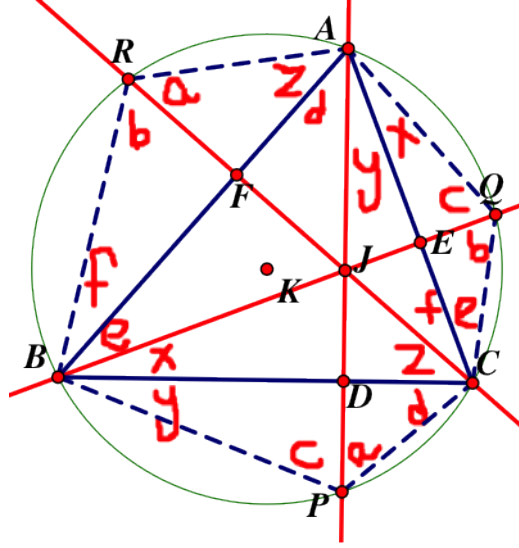
Hence, if we could show $DP = DH$, $EQ = EH$, and $FR = FH$, we would be finished (by our previous result). Proving $DP = DH$ is a matter of proving that the distance from the orthocenter to the foot of the altitude is equal to the distance from the foot to the circumcircle. We can prove this by considering the diagram on the following page, which appears to include three kites ($ARBH$, $CHBP$, and $CQAH$ in particular).



If these are indeed kites, then we have 9 pairs of congruent triangles that would enable to prove $DP = DH$, $EQ = EH$, and $FR = FH$.

Let us consider specifically $\triangle CPD$ & $\triangle CHD$. We know, by the definition of an altitude, that $\angle CDP = 90^\circ = \angle CDH \Rightarrow \angle CDP \cong \angle CDH$. In addition, we have $\overline{CD} \cong \overline{CD}$ by the reflexive property. In order to prove $\triangle CPD \cong \triangle CHD$ and thus that $DP = DH$, we need only show that $\angle HCD \cong \angle PCD$.

By looking at arcs that subtend multiple inscribed angles, we can identify the following:



where $a, b, c, d, e, f, x, y, z$ are angle measures. Looking at our angles $a, d, \& z$, we know by the Exterior Angle Theorem that

$$a + z = 90^\circ = d + a \Rightarrow z = d$$

In particular, $\angle CDH \cong \angle CDP$.

Therefore, by the Angle Side Angle Congruence Theorem, we have $\triangle CPD \cong \triangle CHD$. Thus, by CPCTC, we conclude that $DP = DH$. Similarly, we can show $EQ = EH$ and $FR = FH$ and our desired result has been proven.

In summary, we changed what we wanted to prove - (★) = 4 - into proving a new expression equal to 1. We then found a similar expression equal to 1 and then proved the two equations equivalent using congruent triangles.